Nonlinear Dirac and diffusion equations in 1+1 dimensions from stochastic considerations

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We generalize the method of obtaining fundamental linear partial differential equations such as the diffusion and Schrödinger equation, the Dirac, and the telegrapher's equation from a simple stochastic consideration to arrive at a certain nonlinear form of these equations. A group classification through a one-parameter group of transformations for two of these equations is also carried out.

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I. INTRODUCTION

It is a remarkable fact that some of the fundamental linear equations of physics, such as the diffusion and Schrödinger equations, the Dirac and telegrapher's equations, and the Maxwell equations can be obtained by setting up a master equation from simple stochastic considerations and a modification thereof [1,2].

The transition to a nonlinear equation from a linear equation through certain transformations is well known. An example is the Cole-Hopf transformation that carries over the linear diffusion equation to the nonlinear Burgers equation. The reverse process of getting a linear diffusion equation from a nonlinear diffusion equation in the form $\partial\phi/\partial t = v^2(\partial^2\phi/\partial v^2)$ has also been studied [3,4] through a nonlinear transformation.

In this paper, we propose to obtain a class of nonlinear equations in a different way by generalizing the method of [1]. The method is simple. The basic inputs can be incorporated from considerations and arguments based on physical reasoning to obtain nonlinear equations rather than arbitrary mathematical transformations. The form of the equations obtained is quite restrictive. However, we do not address the deep mathematical significance of the Cole-Hopf transformation and the like in this method.

In Sec. II, we briefly review the method used in setting up some basic linear equations of physics. Then we generalize the procedure of obtaining classes of corresponding nonlinear partial differential equations in Sec. III. Next, Sec. IV is devoted to the construction of the groups under which two of the equations obtained in Sec. II, namely the diffusion equation with nonlinearity and the nonlinear telegrapher's equation, remain invariant. The similarity transformation and the Lie algebra are constructed to show the transformations under which solutions go over to new solutions.

II. LINEAR EQUATIONS

The above-mentioned linear equations have been obtained by Gaveau *et al.* [1] and Ord [2] from stochastic consideration by setting up a master equation. Ord [2] has also arrived at the Maxwell equation in (1+1) dimensions by a modification of the master equation.

Following [1,2], we briefly review how these equations are achieved and then proceed to nonlinear generalization. The basic consideration is the correlation over a random ensemble of particles. However, for simpler visualization we may follow the Boltzmann approach by analyzing the movement of a single particle. Let a particle have random motion in one space dimension moving with a fixed speed v. We assume that it has complete reversal of direction of motion in a random manner from time to time, say as with the flip of a coin. So this is in accord with Poisson distribution, that is to say, there is a fixed rate a for this reversal and the probability for reversal in a time interval dt is adt. Let $P_+(x,t)$ $[P_-(x,t)]$ be the probability density for the particle being at x at time t and moving to the right (left). The master equation for an infinitesimal time step is

$$P_\pm(x,t+\Delta t)=P_\pm(x\mp\Delta x,t)(1-a\Delta t)+P_\mp(x\pm\Delta x,t)a\Delta t. \tag{1}$$

This equation gives rise to the linear equations such as the Dirac, telegrapher's, diffusion, or Schrödinger equations in the lowest approximation under various circumstances.

To the lowest order in Δx and Δt, Eq. (1) gives

$$\frac{\partial P_\pm}{\partial t}=-a(P_\pm-P_\mp)\mp v\frac{\partial P_\pm}{\partial x},\quad v=\frac{\Delta x}{\Delta t}, \tag{2}$$

and the telegrapher's equation follows by iteration,

$$\frac{\partial^2 P_\pm}{\partial t^2}-v^2\frac{\partial^2 P_\pm}{\partial x^2}=-2a\frac{\partial P_\pm}{\partial t}. \tag{3}$$

The one-dimensional Dirac equation is obtained from Eq. (1) by analytic continuation. First we identify $P_\pm$ with $u_\pm$, $v\leftrightarrow c$, $imc^2/\hbar\leftrightarrow a$, and then perform a phase transformation $u(x,t)=e^{(imc^2t/\hbar)}\Psi(x,t)$. This results in

$$i\hbar\frac{\partial\Psi}{\partial t}=mc^2\sigma_x\Psi-ic\hbar\sigma_z\frac{\partial\Psi}{\partial x}. \tag{4}$$

In the Feynman path-integral formulation through checkers moves on space time, 1 has to be replaced by a factor $1+(imc^2/\hbar)\Delta t$ for each step on which a reversal does not take place, whereas for reversals there is a factor $-i\Delta t(mc^2/\hbar)$.

The Dirac equation in (1+1) dimensions, having two components, has a similar time and space dependence to this stochastic approach. But for a scalar object we find that the linear diffusion equation results, which shows the asymmetry in derivatives arising out of the random-walk problem.

A generalization to three space dimensions has been carried out in [1].

McKeon and Ord [5] have shown that if movements backward and forward in time are as well superposed on the previous motion, then the Dirac equation in one dimension results without recourse to direct analytic continuation.

To arrive at the linear diffusion equation in a simple way, we set $P_{\pm}=P_{\mp}=P$ and $a=1/2\Delta t$. The master equation (1) reduces to

$$P(x,t+\Delta t)=\frac{1}{2}P(x-\Delta x,t)+\frac{1}{2}P(x+\Delta x,t). \tag{5}$$

Expanding this in a Taylor series about the point (x,t) gives

$$P(x,t)+\frac{\partial P(x,t)}{\partial t}\Delta t+\cdots=P(x,t)+\frac{\partial^2 P(x,t)}{\partial x^2}\frac{(\Delta x)^2}{2}+\cdots \tag{6}$$

and equating the lowest-order terms we get

$$\frac{\partial P}{\partial t}=\frac{\partial^2 P}{\partial x^2}\frac{(\Delta x)^2}{2\Delta t}=D\frac{\partial^2 P}{\partial x^2}, \tag{7}$$

where $D=(\Delta x)^2/2\Delta t$.

It may be noted that the above equation in the context of Brownian motion can be obtained from a consideration of a one-dimensional random walk with a Bernoulli distribution of probability and the statistical considerations sets [6],

$$\lim_{\Delta t\to 0}\frac{(\Delta x)^2}{2\Delta t}=D, \tag{8}$$

where D is a constant.

A formal analytic continuation (e.g., $t\to it$ or $D\to i\hbar$) leads to the Schrödinger equation for free particles. A potential $V(x,t)$ can be included by adding a term $V(x,t)P(x,t)\Delta t$ to the right-hand side of Eq. (5).

Ord [2] has obtained the Maxwell equations in $1+1$ dimensions by a modification of the master equation. We follow his procedure to show how it is done. First Eq. (1) is modified to

$$P_{\pm}(x,t+\Delta t)=P_{\pm}(x\mp\Delta x,t)+a(x,t)\Delta t, \tag{9}$$

where $a(x,t)$ is interpreted as a source, and linear combinations of P_+ and P_- will correspond to the potentials $A(x,t)$ and $\Phi(x,t)$. To the lowest order in Δx and Δt, Eq. (9) gives

$$\frac{\partial P_{\pm}(x,t)}{\partial t}\Delta t=\mp\frac{\partial P_{\pm}}{\partial x}\Delta x+a(x,t)\Delta t. \tag{10}$$

Writing

$$\begin{aligned} A(x,t)&=\frac{1}{2}[P_+(x,t)+P_-(x,t)],\\ \Phi(x,t)&=\frac{1}{2}[P_+(x,t)-P_-(x,t)], \end{aligned} \tag{11}$$

Eq. (10) implies

$$\frac{\partial A(x,t)}{\partial t}=-c\frac{\partial \Phi(x,t)}{\partial x}+a(x,t), \tag{12}$$

$$\frac{\partial \Phi(x,t)}{\partial t}=-c\frac{\partial A(x,t)}{\partial x}, \tag{13}$$

where we have set $\Delta x/\Delta t=c$.

Equations (10) and (11) may be decoupled by differentiating the first with respect to t and the second with respect to x to give

$$\frac{\partial^2 A(x,t)}{\partial t^2}=c^2\frac{\partial^2 A(x,t)}{\partial x^2}+\frac{\partial a(x,t)}{\partial t} \tag{14}$$

and similarly we get

$$\frac{\partial^2 \Phi(x,t)}{\partial t^2}=c^2\frac{\partial^2 \Phi(x,t)}{\partial x^2}-c\frac{\partial a(x,t)}{\partial t}. \tag{15}$$

Equations (13), (14), and (15) are equivalent to Maxwell equations in (1 + 1) dimensions, Eq. (13) being the Lorentz condition

$$\frac{\partial A(x,t)}{\partial x}+\frac{1}{c}\frac{\partial \Phi(x,t)}{\partial t}=0. \tag{16}$$

In order to obtain the wave equation for the ‘‘vector potential’’ A, we write

$$\frac{1}{c}\frac{\partial a(x,t)}{\partial t}=4\pi J(x,t) \tag{17}$$

and Eq. (14) becomes

$$\frac{\partial^2 A(x,t)}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 A(x,t)}{\partial t^2}=-\frac{4\pi}{c}J(x,t), \tag{18}$$

and similarly writing

$$\frac{1}{c}\frac{\partial a(x,t)}{\partial x}=-4\pi\rho(x,t), \tag{19}$$

Eq. (15) becomes the wave equation for the scalar potential $\Phi(x,t)$,

$$\frac{\partial^2 \Phi(x,t)}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 \Phi(x,t)}{\partial t^2}=-4\pi\rho(x,t). \tag{20}$$

The two definitions (17) and (19) imply that

$$\frac{\partial J(x,t)}{\partial x}+\frac{\partial \rho(x,t)}{\partial t}=0, \tag{21}$$

which is the equation of continuity.

The objective of the above long review is to stress the interesting fact that many of the fundamental linear equations of physics are obtainable from an elementary consideration of stochastic process. Of course, by no stretch of the imagination would we expect the whole of physics to follow from such a consideration.

III. NONLINEAR EQUATIONS

The nonlinear diffusion equation in the form (in our notation)

$$\frac{\partial P(x,t)}{\partial t}=\frac{\partial}{\partial x}\left[f(P)\frac{\partial P(x,t)}{\partial x}\right] \tag{22}$$

is well known in the literature [7,8,4,9] and the properties of its solutions have been extensively studied.

Now we proceed with an aim at getting the above nonlinear equation and others out of the master equation (1) by suitable modifications. If we consider this to be a phenomenological equation, without any recourse to Poisson's distribution, then the obvious way to introduce nonlinearity is to introduce functions of x and t as multiplicative coefficients on the right-hand side of Eq. (1).

Perhaps it would be simplest to replace Δx by $P(x,t)\Delta x$ in Eq. (5), with the resulting equation being

$$\frac{\partial P(x,t)}{\partial t}=DP^2(x,t)\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{23}$$

where

$$D=\frac{(\Delta x)^2}{\Delta t}. \tag{24}$$

Or else, we may treat x and t in the same way, that is, set $\Delta t\rightarrow P(x,t)\Delta t$ and $\Delta x\rightarrow P(x,t)\Delta x$ instead of only $\Delta x\rightarrow P(x,t)\Delta x$, and we get

$$\frac{\partial P(x,t)}{\partial t}=DP(x,t)\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{25}$$

where both Eqs. (23) and (25) are nonlinear equations. Henceforth we set $D=1$.

It should be noted that this does not mean that any nonlinear equation can be obtained in this way. The condition that for $\Delta t=0$ and $\Delta x=0$ both the left and right side of Eq. (1) must match is quite a restriction. However, by making use of Eq. (7) one may get many more equations by setting the source term as a function of x, P and its derivatives or their combinations. This would be analogous to adding terms to the Lagrangian arbitrarily in the conventional method of getting equations of motion.

We also see that if the master Eq. (1) is modified in the first term of the right-hand side as

$$\begin{aligned} &P_{\pm}(x,t+\Delta t)\\ &\quad =P_{\pm}(x\mp\Delta x,t)(1-P_{+}\Delta t)+P_{\mp}(x\pm\Delta x,t)a\Delta t, \end{aligned} \tag{26}$$

we get a nonlinear form of the Dirac equation in one space dimension,

$$\frac{\partial P_{+}}{\partial t}=-P_{+}^2-v\frac{\partial P_{+}}{\partial x}+aP_{-}, \tag{27}$$

$$\frac{\partial P_{-}}{\partial t}=-P_{+}P_{-}+aP_{+}+v\frac{\partial P_{-}}{\partial x}, \tag{28}$$

and by iteration, a nonlinear analog of the telegraphers' equation results,

$$\frac{\partial^2 P_{+}}{\partial t^2}-v^2\frac{\partial^2 P_{+}}{\partial x^2}=-P_{+}^3+P_{+}\frac{\partial P_{+}}{\partial t}+vP_{+}\frac{\partial P_{+}}{\partial x}+a^2P_{+}. \tag{29}$$

Further generalizations would be to consider P as a complex multicomponent object and readers may amuse themselves by putting objects such as supersymmetric variables, Pauli and other matrices, etc., as coefficients of Δx in Eq. (1).

The physical interpretation of P may no longer be the simple probability that it was in the original master equation. This may correspond to some appropriate physical attribute for motion in an inhomogeneous dielectric, viscous medium, or the trajectory in a ''graded'' index optical fiber, for example. We expect in this case the step Δx to depend on the position x where the step is to be taken. Hence one may multiply Δx by an appropriate function of x, $P(x)$ being the simplest choice for the function in our first example. Another analogy that comes to mind is the replacement of the metric $\eta_{\mu\nu}$ by $g_{\mu\nu}(x)$ in general relativity. However, in our case the replacement is in the underlying space itself and is intriguing.

IV. GROUP ANALYSIS

Equations of the form (23) have been analyzed by Munier *et al.* [3] and by Hill [4] in detail. It is found that the nonlinear diffusion equation of the form

$$\frac{\partial \psi}{\partial t}=\psi^2\frac{\partial^2 \psi}{\partial P^2} \tag{30}$$

is equivalent to the classical diffusion equation for P,

$$\frac{\partial P}{\partial t}=\frac{\partial^2 P}{\partial x^2}, \tag{31}$$

if we introduce x such that

$$\psi(P,t)\equiv\frac{\partial P}{\partial x}, \tag{32}$$

and every nonlinear diffusion equation of the form

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial x}\left[f(P)\frac{\partial P}{\partial x}\right] \tag{33}$$

can be transformed to the following equation with a simpler nonlinearity:

$$f(P)\frac{\partial \psi}{\partial t}=\psi^2\frac{\partial^2 \psi}{\partial P^2}, \tag{34}$$

where $\psi(P,t)$ is the flux associated with Eq. (33). Hence, for this special case the analysis would be similar to that of the linear diffusion equation.

However, in general the simplest nonlinear equation that we would get from the master equation by replacing $\Delta x \to f(P)\Delta x$ in Eq. (5) would be

$$\frac{\partial P}{\partial t}=f^2(P)\frac{\partial^2 P}{\partial x^2} \tag{35}$$

as in Eq. (23) or

$$\frac{\partial P}{\partial t}=f(P)\frac{\partial^2 P}{\partial x^2} \tag{36}$$

as in Eq. (25).

Now we proceed to analyze the properties of the solutions of Eq. (35) by means of one-parameter groups as in [7,8,4,9]. For the single dependent variable P and for the two independent variables x and t, we have one-parameter groups of the form

$$\begin{aligned} x_1&=f(x,t,P,\epsilon)=x+\epsilon\xi(x,t,P)+O(\epsilon^2),\\ t_1&=g(x,t,P,\epsilon)=t+\epsilon\eta(x,t,P)+O(\epsilon^2),\\ P_1&=h(x,t,P,\epsilon)=P+\epsilon\zeta(x,t,P)+O(\epsilon^2). \end{aligned} \tag{37}$$

We follow the standard procedure [4,9] to obtain the similarity variable and functional form of the solution by solving the first-order partial differential equation

$$\xi(x,t,P)\frac{\partial P}{\partial x}+\eta(x,t,P)\frac{\partial P}{\partial t}=\zeta(x,t,P) \tag{38}$$

for known functions $\xi(x,t,P)$, $\eta(x,t,P)$, and $\zeta(x,t,P)$. Let

$$\mathbf{v}=\xi(x,t,P)\frac{\partial}{\partial x}+\eta(x,t,P)\frac{\partial}{\partial t}+\zeta(x,t,P)\frac{\partial}{\partial P} \tag{39}$$

be a vector field on the space $X\times U^{(2)}$, where coordinates represent the independent variables, the dependent variables, and the derivatives of the dependent variables up to order 2. All possible coefficient functions ξ, η, ζ are to be determined so that the one-parameter group $\exp(\epsilon\mathbf{v})$ thus obtained would be the symmetry group of the nonlinear equations (35) for the diffusion case and Eq. (29) for the telegrapher's case.

The determining equations for the symmetry group for the diffusion with nonlinearity, Eq. (35), are

Monomial	Coefficients	
$\frac{\partial^2 P}{\partial x\partial t}\frac{\partial P}{\partial x}$	$\eta_P=0$	(A)
$\frac{\partial^2 P}{\partial x\partial t}$	$\eta_x=0$	(B)
$\left(\frac{\partial P}{\partial x}\right)^3$	$\xi_{PP}=0$	(C)
$\left(\frac{\partial P}{\partial x}\right)^2\frac{\partial P}{\partial t}$	$\eta_{PP}=0$	(D)
$\left(\frac{\partial P}{\partial x}\right)^2$	$(\zeta_P-2\xi_x)_P=0$	(E)
$\left(\frac{\partial P}{\partial t}\right)^2$	$-\eta_P+\eta_P=0$	(F)
$\left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial P}{\partial t}\right)$	$-\xi_P=-2\eta_{xP}f(P)-3\xi_P$	(G)
$\frac{\partial P}{\partial x}$	$-\xi_t=f(P)(2\zeta_{xP}-\xi_{xx})$	(H)
$\frac{\partial P}{\partial t}$	$\eta_t=f(P)\eta_{xx}+2\xi_x+\frac{f'(P)\xi}{f(P)}$	(I)
P^0	$\zeta_t-f(P)\zeta_{xx}=0$	(J)

where the prime denotes differentiation with respect to the argument and subscripts denote differentiation with respect to the indicated variable. These equations turn out to be the same as those of the nonlinear diffusion equation of the form

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial x}\left[f(P)\frac{\partial P}{\partial x}\right] \tag{40}$$

considered by Hill [4].

From monomials (A), (B), and (G) it is easily seen that

$$\xi=\xi(x,t),\qquad \eta=\eta(t), \tag{41}$$

$$\zeta_P=2\xi_x+r, \tag{42}$$

where r is a constant. So

$$\zeta_{PP}=0. \tag{43}$$

From monomial (I) we get

$$\zeta=\frac{f(P)}{f'(P)}[2\xi_x-\eta_t] \tag{44}$$

so that either

$$2\frac{\partial\xi}{\partial x}=\frac{\partial\eta}{\partial t} \tag{45}$$

or

$$\left[\frac{f(P)}{f'(P)}\right]_{PP}=0, \tag{46}$$

that is,

$$f(P)=a(P+b)^m, \tag{47}$$

where a, b, and m denote arbitrary constants. If Eq. (45) holds, then from monomial (H) and Eq. (44) we obtain

$$\begin{aligned}\xi(x,t,P)&=\beta+\gamma x,\\ \eta(x,t,P)&=2\theta+2\gamma t,\\ \zeta(x,t,P)&=0,\end{aligned} \tag{48}$$

where β, θ, and γ are arbitrary constants.

Hence, the Lie algebra of infinitesimal symmetries of the equation is spanned by the three vector fields

$$\begin{aligned}\mathbf{v}_1&=\frac{\partial}{\partial x},\\ \mathbf{v}_2&=\frac{\partial}{\partial t},\\ \mathbf{v}_3&=x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t},\end{aligned} \tag{49}$$

and the commutation relations are given by

$$[\mathbf{v}_1,\mathbf{v}_2]=0,\quad [\mathbf{v}_1,\mathbf{v}_3]=\mathbf{v}_1,\quad [\mathbf{v}_2,\mathbf{v}_3]=2\mathbf{v}_2. \tag{50}$$

The one-parameter groups G_i generated by the $\mathbf{v}_i$ are given below. The entries give the transformed points $\exp(\epsilon\mathbf{v}_i)(x,t,P)=(x_1,t_1,P_1)$,

$$\begin{aligned}&G_1:(x+\epsilon,t,P),\\ &G_2:(x,t+\epsilon,P),\\ &G_3:(e^{\epsilon}x,e^{2\epsilon}t,P).\end{aligned} \tag{51}$$

Each group G_i is a symmetry group, and if $P=q(x,t)$ is a solution of our nonlinear diffusion equation, so are the functions

$$\begin{aligned}P^{(1)}&=q(x-\epsilon,t),\\ P^{(2)}&=q(x,t-\epsilon),\\ P^{(3)}&=q(e^{-\epsilon}x,e^{-2\epsilon}t).\end{aligned} \tag{52}$$

The groups we obtain are the same as those for Eq. (40) and so is the similarity variable [4],

$$\omega=\frac{x+\alpha}{(t+\beta)^{1/2}}. \tag{53}$$

However, the functional form

$$P=s(\omega) \tag{54}$$

of the solution satisfies the ordinary differential equation

$$2f(s)\frac{d^2s}{d\omega^2}+\omega\frac{ds}{d\omega}=0, \tag{55}$$

whereas that corresponding to Eq. (40) is given by

$$2f(s)\frac{d^2s}{d\omega^2}+2\frac{df(s)}{ds}\left(\frac{ds}{d\omega}\right)^2+\omega\frac{ds}{d\omega}=0. \tag{56}$$

In the case in which $f(P)$ is given by Eq. (46),

$$\zeta=\left(\frac{P+b}{m}\right)\left[2\frac{\partial\xi}{\partial x}-\frac{\partial\eta}{\partial t}\right], \tag{57}$$

and for the time derivative of ξ we get

$$\frac{\partial\xi}{\partial t}=f(P)\left[1-\frac{4}{m}\right]\frac{\partial^2\xi}{\partial x^2}, \tag{58}$$

while substituting Eq. (57) into monomial (J) and using Eq. (58) gives

$$\eta_{tt}=-\frac{8}{m}\xi_{xxx}. \tag{59}$$

So there are two possibilities arising out of Eq. (58), either for all constants m,

$$\frac{\partial\xi}{\partial t}=\frac{\partial^2\xi}{\partial x^2}=\frac{\partial^2\eta}{\partial t^2}=0, \tag{60}$$

or for $m=4$,

$$\frac{\partial\xi}{\partial t}=\frac{\partial^3\xi}{\partial x^3}=\frac{\partial^2\eta}{\partial t^2}=0. \tag{61}$$

Thus for all m we have

$$\begin{aligned}\xi(x,t,P)&=\mu+\sigma x,\\ \eta(x,t,P)&=\nu+\rho t,\\ \zeta(x,t,P)&=\left(\frac{P+b}{m}\right)(2\sigma-\rho),\end{aligned} \tag{62}$$

where μ, ν, σ, and ρ are arbitrary constants and the infinitesimal symmetries are spanned by four vector fields,

$$\begin{aligned}\mathbf{v}_1&=\frac{\partial}{\partial x},\\ \mathbf{v}_2&=\frac{\partial}{\partial t},\\ \mathbf{v}_3&=x\frac{\partial}{\partial x}+\frac{2}{m}(P+b)\frac{\partial}{\partial P},\\ \mathbf{v}_4&=t\frac{\partial}{\partial t}-\frac{(P+b)}{m}\frac{\partial}{\partial P},\end{aligned} \tag{63}$$

and the commutation relations are given by

$$[\mathbf{v}_1,\mathbf{v}_2]=[\mathbf{v}_1,\mathbf{v}_4]=[\mathbf{v}_2,\mathbf{v}_3]=[\mathbf{v}_3,\mathbf{v}_4]=0,$$
$$[\mathbf{v}_1,\mathbf{v}_3]=\mathbf{v}_1,\quad [\mathbf{v}_2,\mathbf{v}_4]=\mathbf{v}_2. \tag{64}$$

The one-parameter groups G_i generated by the $\mathbf{v}_i$ are

$$G_1:(x+\epsilon,t,P),$$
$$G_2:(x,t+\epsilon,P),$$
$$G_3:(e^{\epsilon}x,t,(P+b)e^{2\epsilon/m}),$$
$$G_4:(x,e^{\epsilon t},(P+b)e^{-\epsilon/m}), \tag{65}$$

and if $P=y(x,t)$ is a solution to our nonlinear diffusion equation, so are the functions

$$P^{(1)}=y(x-\epsilon,t,P),$$
$$P^{(2)}=y(x,t-\epsilon,P),$$
$$P^{(3)}=y(e^{-\epsilon}x,t,(P-b)e^{-2\epsilon/m}),$$
$$P^{(4)}=y(x,e^{-\epsilon}t,(P-b)e^{\epsilon/m}). \tag{66}$$

The similarity variable in this case is given by

$$\omega=\frac{x+\dfrac{\mu}{\sigma}}{\left(t+\dfrac{\nu}{\rho}\right)^{\sigma/\rho}} \tag{67}$$

and the functional form of the solution is

$$P=\left(t+\frac{\nu}{\rho}\right)^{[(2\sigma/m\rho)-1]}s(\omega)-b. \tag{68}$$

Now for the nonlinear form of the telegrapher's equation (29), arising out of the nonlinear Dirac equation (27), the independent determining equations of the symmetry group are given below.

Monomial	Coefficient	
$\dfrac{\partial^2 P_+}{\partial t^2}\dfrac{\partial P_+}{\partial t}$	$\eta_{P_+}=0$	(a)
$\dfrac{\partial^2 P_+}{\partial x\partial t}\dfrac{\partial P_+}{\partial t}$	$\xi_{P_+}=0$	(b)
$\dfrac{\partial^2 P_+}{\partial x\partial t}$	$\xi_t=v^2\eta_x$	(c)
$\dfrac{\partial^2 P_+}{\partial t^2}$	$\eta_t=\xi_x$	(d)
$\dfrac{\partial P_+}{\partial t}$	$2\zeta_{tP_+}-\eta_{tt}+v^2\eta_{xx}+P_+(\zeta_{P_+}-2\xi_x)-P_+v\,\eta_x-P_+\eta_x=0$	(e)
$\dfrac{\partial P_+}{\partial x}$	$\xi_{tt}+v^2(2\zeta_{xP_+}-\xi_{xx})+\zeta+P_+v(\zeta_{P_+}-2\xi_x)+P_+(\zeta_{P_+}-\xi_x)+vP_+(\zeta_{P_+}-\xi_x)=0$	(f)
$(P_+)^0$	$\zeta_{tt}-v^2\zeta_{xx}-P_+^3(\zeta_{P_+}-2\xi_x)+P_+a^2(\zeta_{P_+}-2\xi_x)+3P_+^2\zeta+P_+\zeta_x+v\,\zeta_{xP_+}+a^2\zeta=0$	(g)

The solutions are given by

$$\xi(x,t,P_+)=Av^2t+B,$$
$$\eta(x,t,P_+)=Ax+E, \tag{69}$$
$$\zeta(x,t,P_+)=0,$$

where A, B, and E are arbitrary constants and the infinitesimal symmetries are spanned by the three vector fields

$$\mathbf{v}_1=\frac{\partial}{\partial x},\quad \text{space translation,}$$
$$\mathbf{v}_2=\frac{\partial}{\partial t},\quad \text{time translation,} \tag{70}$$
$$\mathbf{v}_3=v^2t\frac{\partial}{\partial x}+x\frac{\partial}{\partial t},\quad \text{hyperbolic ``rotation'' in } (x,t) \text{ space,}$$

with the commutation relations

$$[\mathbf{v}_1,\mathbf{v}_2]=0,\quad [\mathbf{v}_1,\mathbf{v}_3]=\mathbf{v}_2,\quad [\mathbf{v}_2,\mathbf{v}_3]=v^2\mathbf{v}_1. \tag{71}$$

The one-parameter groups G_i generated by the $\mathbf{v}_i$ are

$$G_1:(x+\epsilon,t,P_+),$$
$$G_2:(x,t+\epsilon,P_+), \tag{72}$$
$$G_3:(x+v^2\epsilon t,t+\epsilon x,P_+).$$

This implies that if $P_+=z(x,t)$ is a solution to Eq. (29), so are the functions

$$P_{+1}=z(x-\epsilon,t),$$
$$P_{+2}=z(x,t-\epsilon), \tag{73}$$
$$P_{+3}=z(x-v^2\epsilon t,t-\epsilon x),$$

where ϵ is any real number.

In order to compare the above vector fields of Eq. (70) with those of the linear second-order form of the telegrapher's equation (3), we have the corresponding independent determining equations of the symmetry group:

Monomial	Coefficient	
$\frac{\partial^2 P_+}{\partial x^2}\frac{\partial P_+}{\partial x}$	$\xi_{P_+}=0$	(P)
$\frac{\partial^2 P_+}{\partial x^2}\frac{\partial P_+}{\partial t}$	$\eta_{P_+}=0$	(Q)
$\frac{\partial^2 P_+}{\partial x^2}$	$\xi_x=\eta_t$	(R)
$\frac{\partial^2 P_+}{\partial x\partial t}$	$\xi_t=v^2\eta_x$	(S)
$\left(\frac{\partial P_+}{pt}\right)^2$	$\zeta_{P_+P_+}-2\eta_{tP_+}+4a\eta_{P_+}=0$	(T)
$\frac{\partial P_+}{\partial x}$	$\xi_{tt}-v^2\xi_{xx}+2v^2\zeta_{xP_+}+2a\zeta_t=0$	(U)
$\frac{\partial P_+}{\partial t}$	$\eta_{tt}-v^2\eta_{xx}-2\zeta_{tP_+}-2a\eta_t=0$	(V)
$(P_+)^0$	$\zeta_{tt}-v^2\zeta_{xx}+2a\zeta_t=0$	(W)

The solutions are given by

$$\xi(x,t,P_+)=Kv^2t+L,$$
$$\eta(x,t,P_+)=Kx+M, \tag{74}$$
$$\zeta(x,t,P_+)=-KaxP_++NP_+,$$

where K, L, M, and N are arbitrary constants. The infinitesimal symmetries are spanned by the four vector fields

$$\mathbf{v}_1=\frac{\partial}{\partial x},$$
$$\mathbf{v}_2=\frac{\partial}{\partial t},$$
$$\mathbf{v}_3=P_+\frac{\partial}{\partial P_+}, \tag{75}$$
$$\mathbf{v}_4=v^2t\frac{\partial}{\partial x}+x\frac{\partial}{\partial t}-axP_+\frac{\partial}{\partial P_+},$$

with commutation relations

$$[\mathbf{v}_1,\mathbf{v}_2]=[\mathbf{v}_1,\mathbf{v}_3]=0,\quad [\mathbf{v}_1,\mathbf{v}_4]=\mathbf{v}_2-a\mathbf{v}_3,$$
$$[\mathbf{v}_2,\mathbf{v}_3]=0,\quad [\mathbf{v}_2,\mathbf{v}_4]=v^2\mathbf{v}_1,\quad [\mathbf{v}_3,\mathbf{v}_4]=0. \tag{76}$$

We have ignored the obvious infinite-dimensional subalgebras in the above analysis.

V. CONCLUSION

The main objective of this paper was to extend the method of deducing some fundamental linear partial differential equations of physics from a consideration of stochastic arguments to the nonlinear case. We saw that this could be achieved in a very simple way by modifying the master equation to obtain the ''nonlinear diffusion'' equation, a ''nonlinear Dirac equation'' in 1+1 dimensions, and the corresponding ''nonlinear telegrapher's equation.'' As a preliminary step towards the analysis of the properties of the solutions, we have considered the group classification problem of the first and the last one by means of one-parameter groups. The infinitesimal symmetry group of the nonlinear telegrapher's equation is spanned by a vector field corresponding to a ''hyperbolic rotation'' of x and t. For our type of diffusion equation, although the group structure is similar to that of the standard nonlinear diffusion equation, the ordinary differential equations obtained are different and the results are similar when $m=4$ in our case, but $m=-\frac{4}{3}$ in the standard case (m being the highest power of the dependent variable in coefficient to the $\partial^2/\partial x^2$ term in the nonlinear diffusion equation). The physical applications of this equation have been widely studied in the context of gas dynamics and plasma physics, etc. We expect the other two equations to have similar important applications in physics with rich mathematical structure and we leave it for future study. However, as a comparison of Eqs. (70) and (75) shows, one does see explicitly which symmetries get broken when the equation is modified.

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